AP Calculus BC

Unit 4 – Contextual Applications of Differentiation

AP Calculus BC – Worksheet 22 **Linearization**

In problems 1 & 2, **(a)** find the linearization $L(x)$ of $f(x)$ at $x=a$. **(b)** How accurate is the approximation $L(a+0.1)$ to the actual value of $f(a+0.1)$?

1) $f(x)=x^3-2x+3, \quad a=2$	2) $f(x)=\sqrt[3]{x}, \quad a=8$

3	Show that the linearization of $f(x)=(1+x)^k$ centered at $x=0$ is $L(x)=1+kx$
4	Approximate $\sqrt{101}$ using a linearization centered at an appropriate nearby number.
5	If $f(x)=x^3+3x$, approximate $f(2.01)$ using linearization centered at $x=2$.
6	Approximate $\sqrt{24.9}+(24.9)^2$ using linearization.
7	For the function *f*, $f'=2x+1$ and $f(1)=4$. What is the approximation for $f(1.2)$ using the tangent line approximation centered at $x=1$?
8	Find an approximate value for $f(-3.9)$ on $f(x)=\sqrt{x^2+9}$ using linearization.
9	Approximate using a tangent line approximation: $(8.4)^{4/3}$.
10	Let *f* be a function such that each point (x,y) on the graph of *f*, the slope is given by $\frac{dy}{dx}=12x-14y^2$. The graph of *f* passes through the point $(1,-2)$ and is concave up on the interval $1<x<1.5$. Determine an approximation for $f(1.3)$. Is this approximation an underestimate or an overestimate? Justify your answer.
11	<table><tr><td>x</td><td>2.8</td><td>3.0</td><td>3.2</td><td>3.4</td></tr><tr><td>$g'(x)$</td><td>1.05</td><td>-1.2</td><td>-0.8</td><td>1.3</td></tr></table> Selected values of the derivative of the function *g* are given in the table above. It is known that $g(3)=17$. What is the approximation for $g(3.2)$ found using the line tangent to the graph of *g* at $x=3$?

AP Calculus BC – Worksheet 23 **Rates of Change**

1

(a) Write the area A of a circle as a function of the circumference C.

(b) Find the (instantaneous) rate of change of the area A with respect to the circumference C.

(c) If C is measured in inches and A is measured in square inches, what units would be appropriate for $\frac{dA}{dC}$?

(d) Evaluate the rate of change of A at $C=\pi$ and $C=6\pi$.

2

The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t)=200(30-t)^2$. How fast is the water running out at the end of 10 minutes? What is the average rate at which the water flows out during the first 10 minutes?

3

The volume $V=\frac{4}{3}\pi r^3$ of a spherical balloon changes with the radius.

(a) At what rate does the volume change with respect to the radius when $r=2$ ft? Include units of measure.

(b) By how much does the volume increase when the radius changes from 2 to 2.2 feet?

4

t (hours)	0	3	8	12	14
$S'(t)$ (centimeters per hour)	2.3	2.1	1.9	1.7	1.6

The depth of snow in a field is given by a twice-differentiable function S for $0 \le t \le 14$, where $S(t)$ is measured in centimeters and time t is measured in hours. Values of $S'(t)$, the derivative of S, at selected values of time t are shown in the table above. It is known that the graph of S is concave down for $0 \le t \le 14$.

(a) Use the data in the table to approximate $S''(10)$. Show the computations that lead to your answer. Using correct units, explain the meaning of $S''(10)$ in the context of the problem.

(b) Is there a time t, for $0 \le t \le 14$, at which the depth of snow is changing at a rate of 2 centimeters per hour? Justify your answer.

(c) At time $t=8$, the depth of snow is 45 centimeters. Use the line tangent to the graph of S at $t=8$ to approximate the depth of snow at time $t=10$. Is the approximation an underestimate or an overestimate of the actual depth of snow at time $t=10$? Justify your answer.

(d) The depth of snow, in centimeters, is also modeled by the twice-differentiable function D for $0 \le t \le 14$, where $D(t)=120-92e^{-\frac{t}{40}}$ and time t is measured in hours. Use the model to find $D'(10)$. Using correct units, explain the meaning of $D'(10)$ in the context of the problem.

5

Suppose that $C(T)$ is the cost of heating your house, in dollars per day, when the temperature outside is T degrees Fahrenheit.

(a) Explain in practical terms the meaning of $C'(23)=-0.21$.

(b) If $C(23)=2.94$ and $C'(23)=-0.21$, approximate the cost to heat your house when the temperature is 20 degrees, $C(20)$. Show the work that leads to your answer.

AP Calculus BC – Worksheet 24 **Straight-Line Motion**

1. A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = -t^3 + 7t^2 - 14t + 8$ where s is measured in meters and t is measured in seconds.
   (a) Find an equation that can be used to find the particle's rate of change at any time t.
   (b) At what rate is the particle moving when $t = 4$ seconds? Indicate proper units. Using this value, identify the direction in which the particle is moving.
   (c) Find the particle's average rate of change over the time interval $[0,12]$. Indicate proper units.
   (d) When is the particle not moving?

2. The height of a rock is given by $h = 3 + 135t - 16t^2$. Find the velocity and acceleration when $t = 3$ seconds. When $t = 6$ seconds.

3. A particle moves along a line by $s(t) = t^2 - 5t - 8$. At $t = 3$,
   a) find the position.
   b) determine the velocity.
   c) Determine the displacement from $t = 1$ to $t = 7$.

4. The coordinates, s, of a moving body for various times t are given in the table below.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	3.5	−4	−8.5	−10	−8.5	−4	3.5	14	27.5

Assuming that the function s is a twice-differentiable function, estimate the velocity at $t = 2.3$ sec.

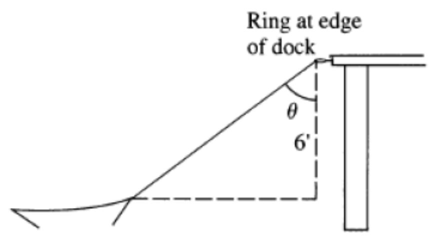
5. (**Calculator Permitted)** A particle moves along a line so that at time t, where $0 \leq t \leq \pi$, its position is given by $s(t) = -4\cos t - \dfrac{t^2}{2} + 10$. What is the velocity of the particle when its acceleration is zero?

   A) -5.19 B) 0.74 C) 1.32 D) 2.55 E) 8.13

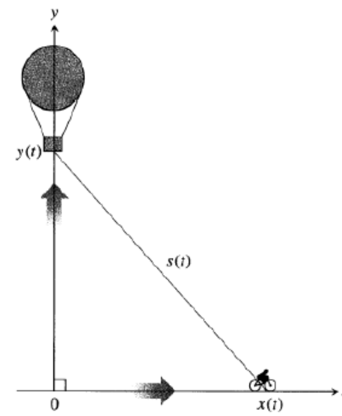
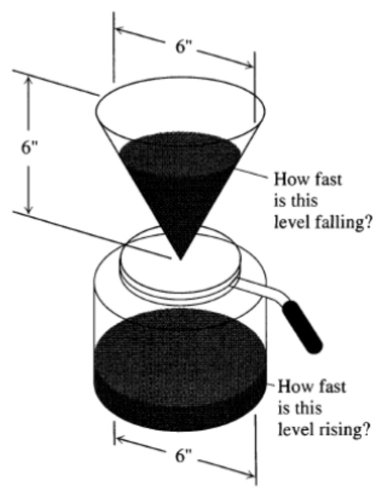
6. (**Calculator Permitted**) A particle moves along a straight line with velocity given by $v(t) = 4 - (0.98)^{-t^2}$ at time $t \geq 0$. What is the acceleration of the particle at time $t = 4$?
   (A) -0.223 (B) 2.618 (C) 8.284 (D) 0.010 (E) -0.092

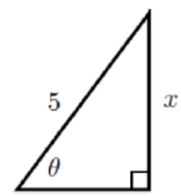
7. Suppose a car travels along a straight road according to the velocity function $v(t) = t^2 - 4$, for $t \in [0,5]$. Where v is measured in miles per hour and t is in hours. Sketch the velocity of the function over the interval. During what times is the car traveling in the negative direction? Positive direction? At what time does the car change directions?

8. A body's velocity at time t seconds is $v = 2t^3 - 9t^2 + 12t - 5$ m/sec. Find the body's speed each time the acceleration is zero.

1	***Changing dimensions in a rectangle.*** The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change of (a) the area (b) the perimeter (c) the lengths of the diagonals of the rectangle.
2	***The radius of an inflating balloon.*** A spherical balloon is inflated with helium at the rate of $100\pi\ \text{ft}^3/\text{min}$. At the instant that the radius of the balloon is 5 feet, (a) how fast is the balloon's radius increasing? (b) How fast is the surface area increasing?
3	***Hauling in a dinghy.*** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 feet above the bow (see figure below). The rope is hauled in at the rate of 2 ft/sec. (a) How fast is the boat approaching the dock when 10 feet of rope are out? (b) At what rate is angle θ changing at the same moment? 
4	A 26-foot ladder leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 4 feet per second. How fast is the top of the ladder sliding down the wall when the foot is 10 feet from the wall?
5	How fast is the radius of a spherical soap bubble changing when air is blown into it at a rate of 10 cubic centimeters per second at the instant the radius is 1 cm?

AP Calculus BC – Worksheet 26 **Related Rates 2**

1	***A draining conical reservoir***. Water is flowing at the rate of $50\ \text{m}^3/\text{min}$ from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. At the instant when the water is 5 m deep, a) How fast is the water level falling? b) How fast is the radius of the water's surface changing?
2	***A balloon and a bicycle***. A balloon is rising vertically above a level, straight road at a constant rate of $1\ \text{ft/sec}$ (see figure at right). Just when the balloon is 65 feet above the ground, a bicycle moving at a constant rate of $17\ \text{ft/sec}$ passes under it. How fast is the distance between the bicycle and balloon changing 3 seconds later? 
3	***Making coffee.*** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $10\ \text{in}^3/\text{min}$ (see figure at right). (a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep? (b) How fast is the level in the cone falling at the same instant described in (a)? 
4	***Motion in the plane***. The coordinates of a particle in the *xy*-plane are differentiable funcitons of time t with $\frac{dx}{dt} = -1\ \text{m/sec}$ and $\frac{dy}{dt} = -5\ \text{m/sec}$. How fast is the particle's distance from the origin changing as it passes through the point $(5,12)$?
5	***A moving shadow*** A 5-foot tall man walks at the rate of 5 ft/sec toward a streetlight that is 16 feet above the ground (a) At what rate is the length of his shadow changing? (b) At what rate is the tip of his shadow moving?

1	A street light is on top of a 10-foot pole. A 6-foot man walks away from the pole at a rate of 4 feet per second. At what rate is the tip of the man's shadow moving from the base of the pole when he is 10 feet from the pole?
2	A point is moving on the graph $5x^3+6y^3=xy$. When the point is at $\left(\frac{1}{11},\frac{1}{11}\right)$, its y-coordinate is increasing at a speed of 5 units per second. What is the rate of change of the x-coordinate at that time and in which direction is the point moving?
3	In the right triangle shown at right, the angle θ is increasing at a constant rate of 2 radians per hour. At what rate is the side length of x increasing when $x=4$ feet? 
4	A container ahs the shape of an open right circular cone, as shown in the figure at right. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constan rate of $-\frac{3}{10}\text{cm/hr}$. a) Find the volume V of water in the container when $h=5$ cm. Indicate units of measure b) Find the rate of change of the volume of what in the container, with respect to time, when $h=5$ cm. Indicate units of measure. c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?
5	Water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base area 400π square feet (see figure at right, not drawn to scale). The depth h, in feet, of the water in the conical tank is changing at the reate $(h-12)$ feet per minute. a) Write an expression fo rhe volume of water in the conical tank as a function of h. b) At what rate is the volume of water in the conical tank changing when $h=3$? Indicate units of measure. c) Let y be the depth, in feet, of water in the cylindrical tank. At what rate is y changing when $h=3$? Indicate units of measure.
6	***Moving along a parabola.*** A particle moves along the parabola $y=x^2$ in the first quadrant in such a way that the x-coordinate (measured in meters) increases at a steady $10\,\text{m/sec}$. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x=3$ m?

AP Calculus BC – Worksheet 28 **L'Hopital's Rule**

Evaluate each limit.

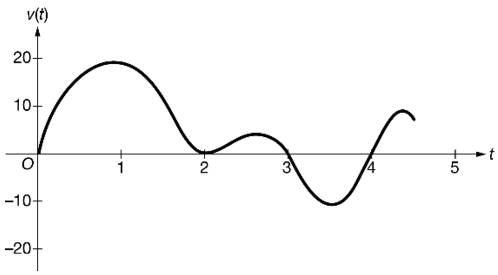
1) $\lim\limits_{x \to 0} \dfrac{x}{\tan x}$	2) $\lim\limits_{x \to 1} \dfrac{1 - 1/x}{1 - 1/x^2}$	3) $\lim\limits_{x \to \infty} \dfrac{\log_2 x}{\log_3 x}$
4) $\lim\limits_{h \to 0} \dfrac{8\left(\frac{1}{2} + h\right)^8 - 8\left(\frac{1}{2}\right)^8}{h}$	5) $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{\tan x}$	6) $\lim\limits_{m \to 0} \dfrac{1}{m} \ln\left(\dfrac{2+m}{2}\right)$
7) $\lim\limits_{n \to \infty} \dfrac{4n^2}{10000n + n^2}$	8) $\lim\limits_{\theta \to 0} \dfrac{\arctan \theta}{2\theta}$	9) $\lim\limits_{v \to \infty} \dfrac{v^2}{e^{-v}}$
10) $\lim\limits_{x \to \pi^+} \dfrac{2x - 2\pi}{\sin(x - \pi)}$	11) $\lim\limits_{x \to \pi} \dfrac{\pi + \sec x}{x^2 - \pi^2}$	12) $\lim\limits_{x \to 0} \dfrac{\sin x}{\ln(2e^x - 1)}$

13) Let f be the function defined by $f(x) = 2x + 3e^{-5x}$, and let g be a differentiable function with derivative given by $g'(x) = \dfrac{1}{x} + 4\cos\left(\dfrac{5}{x}\right)$. It is known that $\lim\limits_{x \to \infty} g(x) = \infty$. Find the value of $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)}$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
3	2	1	-1	-2

14) Selected values of the twice-differentiable functions f and g and their derivatives are given in the table above. Find the value of $\lim\limits_{x \to 3} \dfrac{xf(x) - 6}{g(x) + 1}$.

AP Calculus BC – Worksheet 29 **Unit 4 Review: Contextual Applications of the Derivative**

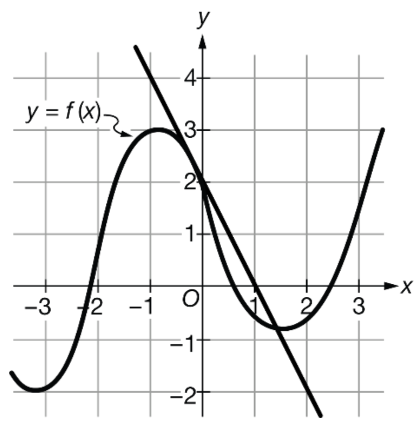
1	If $f(x)=2x^2-x$, approximate $f(3.1)$ using linearization centered at $a=3$.
2	The function $Q(t)=3e^{-0.7t}\sin\left(\frac{t}{3}\right)+0.01\sin(4t)-0.02\cos(4t)$ models the electric charge, measured in coulombs, inside a lightbulb t seconds after it is turned on. Explain what is meant, in the context of the problem by $Q'(4)=-0.171$.
3	The function $t=f(A)$ models the time, in minutes, for a chemical reaction to occur as a function of the amount A of catalyst used, measured in milliliters. What are the units for $f''(A)$?
4	 A particle moves along the *x*-axis. The graph of the particle's velocity $v(t)$ at time t is shown above for $0<t<4.5$. a) Determine the open interval(s) on which the particle is moving to the right. Explain your reasoning. b) At which values of t does the particle change direction? Explain your reasoning.
5	A particle moves along the *y*-axis so that at time $t\ge 0$ its position is given by $y(t)=\frac{2}{3}t^3-5t^2+8t$. Over the time interval $0<t<5$, for what values of t is the speed of the particle increasing?
6	A triangle has base *b* centimeters and height *h* centimeters, where the height is three times the base. Both *b* and *h* are functions of time *t*, measured in seconds. If *A* represents the area of the triangle, determine an expression that gives the rate of change of *A* with respect to *t*.
7	A particle moves on the circle $x^2+y^2=100$ in the *xy*-plane for time $t\ge 0$. At the time when the particle is at the point $(8,6)$, the *x*-value is changing at a rate of 5 units/second. What is the rate of change of the *y*-coordinate?
8	A 10-foot ladder is leaning straight up against a wall when a person begins pulling the base of the ladder away from the wall at the rate of 1 foot per second. Determine the rate at which the top of the ladder is sliding down the wall when the base of the ladder is 9 feet from the wall.
9	The line tangent to the graph of the twice-differentiable function *f* at the point $x=3$ is used to approximate the value of $f(3.25)$. Explain how the tangent line approximation would be guaranteed to be an underestimate of $f(3.25)$.
10	Evaluate each limit: a) $\lim\limits_{x\to 0}\frac{4x^2}{\cos(x)-1}$ b) $\lim\limits_{x\to 3}\frac{\ln\left(\frac{x}{3}\right)}{x^2-7x+12}$ c) $\lim\limits_{x\to \pi}\frac{\pi-x}{\sin(2x)-1}$ d) $\lim\limits_{x\to \infty}\frac{x^{10}}{e^{2x}+x}$

11

x	$f(x)$	$f(x)$	$g(x)$	$g'(x)$
2	4	3	2	1

Selected values of the twice-differentiable function *f* and *g* and their derivatives are given in the table above. Find the value of $\lim\limits_{x \to 2} \dfrac{x^2 f(x) - 16}{g(x) - 2}$.

12



The figure above shows the graph of the twice-differentiable function *f* and the line that is tangent to the graph of *f* at the point $(0, 2)$. Find the value of $\lim\limits_{x \to 0} \dfrac{f(x)e^{-x} - 2}{x^2 - 2x}$.

A graphing calculator is required for this question.

1. For time $0 \le t \le 8$, people arrive at a venue for an outdoor concert at a rate modeled by the function *A* defined by $A(t) = 0.3\sin(1.9t) + 0.3\cos(0.6t) + 1.3$. For time $0 \le t \le 1$, no one leaves the venue, and for time $1 \le t \le 8$, people leave the venue at a rate modeled by the function *L* defined by $L(t) = 0.2\cos(1.9t) + 0.2\sin t + 0.8$. Both $A(t)$ and $L(t)$ are measured in hundreds of people per hour, and *t* is measured in hours. The number of people at the venue, in hundreds, at time *t* hours is given by $P(t)$.
   (a) At time $t = 2$, there are 6 hundred people at the venue. Write an equation for the locally linear approximation of *P* at $t = 2$, and use it to approximate the number of people at the venue at time $t = 2.5$.
   (b) Find $P''(5)$. Using correct units, interpret the meaning $P''(5)$ in the context of the problem.
   (c) Is there a time *t*, for $1 < t < 8$, at which the rate of change of the number of people at the venue changes from negative to positive? Give a reason for your answer.
   (d) A rectangular "standing-only" section at the venue changes size as *t* increases in order to manage the flow of people. Let *x* represent the length, in feet, of the section, and let *y* represent the width, in feet, of the section. The length of the section is increasing at a rate of 6 feet per hour, and the width of the section is decreasing at a rate of 3 feet per hour. What is the rate of change of the area with respect to time when $x = 16$ and $y = 10$? Indicate units of measure.

No calculator is allowed for this question.

$$W(t) = \begin{cases} \frac{32}{5} + \frac{2}{5}\cos\left(\frac{\pi t}{4}\right) & \text{for } 0 \le t \le 4 \\ 6 + \frac{1}{6}(t-4)^2 & \text{for } 4 < t \le 9 \end{cases}$$

2. The depth of water in a zoo aquarium at a certain point is modeled by the function *W* defined above, where $W(t)$ is measured in feet and time *t* is measured in hours.
   (a) Find $W'(7)$. Using correct units, explain the meaning of $W'(7)$ in the context of the problem
   (b) The graph of *W* is concave up for $2 \le t \le 2.5$. Use the line tangent to the graph of *W* at $t = 2$ to show that $W(2.5) \ge 6$.
   (c) Find $\lim\limits_{t \to 2} \dfrac{W(t) - t^2 - \frac{12}{5}}{t - 2}$.

No calculator is allowed for this question.

3. Particle P moves along the x-axis so that its position at time $t > 0$ is given by $x_P(t) = \dfrac{e^{2-t} - 2t}{e^{2-t} + 3t}$. A second particle, particle Q, also moves along the x-axis so that its position at time t is given $x_Q(t) = t^3 - 3t^2 + 5$.

   (a) Show that the velocity of particle P at time t is given by $v_P(t) = \dfrac{-5te^{2-t} - 5e^{2-t}}{\left(e^{2-t} + 3t\right)^2}$.

   (b) At time $t = 2$, particle Q is at rest. At time $t = 2$, is particle P moving toward particle Q or away from particle Q? Explain your reasoning.

   (c) The acceleration of particle Q is given by $a_Q(2)$. Find the value of $a_Q(2)$.

   (d) Describe the position of particle P and the position of particle Q as t approaches infinity. Show the work that leads to your answers.